Asymptotic Expansion of Integrals

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Abstract

These notes are largely based on the last 3 weeks of **Math 6720: Applied Complex Variables and Asymptotic Methods** course, taught by Christel Hohenegger in Spring 2017 and Alexander Balk in Spring 2016, at the University of Utah. Additional examples/remarks/results from other sources are added as I see fit, purely for my own understanding. These notes are by no means accurate or applicable, and any mistakes here are of course my own. Please report any typographical errors or mathematical fallacy to me by email tan@math.utah.edu

Motivation

A solid understanding in the asymptotic theory of integrals has proven to be invaluable for applied mathematician, but why integrals? The reason is that many functions that arise frequently in mathematics, physics and engineering are defined by (complicated) integral expressions, and in most cases one resorts to numerical techniques to study these integrals due to the difficulty in gauging its behaviour. It is precisely this reason that many powerful analytical tools are developed to extract asymptotic behaviour of these integral functions for small or large values of the parameter. We briefly mention a few applications:

1. **Integral transforms.** The Fourier transform of a given function $f(x)$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi x\cdot\xi}\, dx,$$

and one is interested in estimating asymptotically $\hat{f}(\xi)$ as $\xi \longrightarrow \infty$. For simple functions such as $e^{-\alpha|x|}$ and e^{-x^2}, one can use contour integration to compute their Fourier transform explicitly. For L^1 functions, Riemann-Lebesgue lemma states that $\hat{f}(\xi) \longrightarrow 0$ as $|\xi| \longrightarrow \infty$. However, these approaches are not available for complicated functions such as bump functions.

2. **Special functions.** Examples of special functions in mathematical physics include:

$$\begin{aligned} \text{Airy function : } \ \mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right) dt \\ \text{Gamma function : } \ \Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\, dt, \quad \mathrm{Re}(z) > 0. \end{aligned}$$

Airy functions appear in optics, electromagnetism, fluid dynamics and nonlinear wave propagation.

3. **Differential equations.** In special cases, one might have an integral representation for solution of ODEs and PDEs. Long time behaviour of the system can be understand using asymptotic expansion techniques. Consider the following initial value problem:

$$\begin{cases} xy'''(x) + 2y = 0, \\ y(0) = 0, \quad y(\infty) = 0. \end{cases}$$

Its solution has an integral representation

$$y(x) = \int_0^\infty \exp\left(-t - \frac{x}{\sqrt{t}}\right) dt.$$

One can show that asymptotically the solution satisfies

$$y(x) \sim \sqrt{\frac{\pi}{3}} \sqrt[3]{4x} \exp\left(-3\left(\frac{x}{2}\right)^{2/3}\right) \quad \text{as } x \longrightarrow \infty.$$

1 Asymptotic Notation

We begin by defining asymptotic notations and asymptotic expansion. These are useful in describing the limiting behaviour of a function when the argument gets closer to a particular complex number, typically 0 or ∞.

Definition 1.1. Let $f(z), \phi(z)$ be functions defined on $\Omega \subset \mathbb{C}$.

1. We say that $f(z) = \mathcal{O}(\phi(z))$ as $z \longrightarrow z_0$ if there exists a constant $C > 0$ and a neighbourhood U of z_0 such that

$$|f(z)| \leqslant C|\phi(z)| \quad \text{for all } z \in \Omega \cap U.$$

i.e. $\left|\dfrac{f(z)}{\phi(z)}\right|$ is bounded locally around z_0.

2. We say that $f(z) = o(\phi(z))$ as $z \longrightarrow z_0$ if

$$\lim_{z \to z_0} \frac{f(z)}{\phi(z)} = 0.$$

3. A sequence of gauge functions $\{\phi_n\}_{n=0}^\infty$ is an **asymptotic sequence** as $z \longrightarrow z_0$ if

$$\phi_{n+1}(z) = o(\phi_n(z)) \quad \text{as } z \longrightarrow z_0 \quad \text{for every } n = 0, 1, \ldots.$$

4. The function $f(z)$ is said to have an **asymptotic representation (expansion)**

$$f(z) \sim f_N(z) = \sum_{n=0}^{N} a_n \phi_n(z) \quad \text{as } z \longrightarrow z_0,$$

if for every $N = 0, 1, 2, \ldots$, we have

$$f(z) - f_N(z) = o(\phi_N(z)) \quad \text{as } z \longrightarrow z_0.$$

In other words, asymptotic representation of a function describes its asymptotic behaviour in terms of asymptotic sequence.

Remark 1.2.

1. Intuitively, an asymptotic expansion of a given function f is a finite sum which might diverges, yet it still provides an increasingly accurate description of the asymptotic behaviour of f. There is a caveat here: for a divergent asymptotic expansion, for some z, there exists an optimal $N_0 = N_0(z)$ that gives best approximation to f, *i.e.* adding more terms actually gives worse accuracy.

2. However, for values of z sufficiently close to the limiting value z_0, the optimal number of terms required increases, *i.e.* for every $\varepsilon > 0$, there exists an δ and an optimal $N_0 = N_0(\delta)$ such that

$$\left| f(z) - \sum_{k=0}^{N} a_k \phi_k(z) \right| < \varepsilon \quad \text{for every } |z - z_0| < \delta \text{ and } N > N_0.$$

3. One should think of $f_N(z)$ as converging for fixed N in the limit as $z \longrightarrow z_0$. Observe that the definition of asymptotic expansion implies that the remainder term is "small" compared to the last term $\phi_N(z)$ of $f_N(z)$.

Example 1.3. The functions $\phi_k(x) = x^k$ form an asymptotic sequence as $x \longrightarrow 0^+$ and in this case the asymptotic representation is often called an **asymptotic power series**. The functions $\phi_k(x) = x^{-k}$ form an asymptotic sequence as $x \longrightarrow \infty$.

Proposition 1.4. *Consider finding the leading asymptotic behaviour of the integral*

$$I(x) = \int_a^b f(x,t)\,dt \quad \text{as } x \longrightarrow x_0.$$

If $f(x,t) \sim f_0(t)$ as $x \longrightarrow x_0$ uniformly for $t \in [a,b]$, i.e.

$$\lim_{x \to x_0} \frac{f(x,t) - f_0(t)}{f_0(t)} = 0 \quad \text{uniformly in } t,$$

then the leading behaviour of $I(x)$ as $x \longrightarrow x_0$ is

$$I(x) = \int_a^b f(x,t)\,dt \sim \int_a^b f_0(t)\,dt \quad \text{as } x \longrightarrow x_0,$$

provided that the integral on the RHS is finite and nonzero.

Example 1.5. For instance, to determine the leading behaviour of the integral

$$I(x) = \int_0^2 \cos\big(xt^2 + x^2 t\big)^{1/3}\,dt \quad \text{as } x \longrightarrow 0,$$

we simply set $x = 0$ and obtain

$$I(x) \sim \int_0^2 \cos(0)\,dt = 2 \quad \text{as } x \longrightarrow 0.$$

2 Series Expansions and Integration By Parts

Broadly speaking, there are two ways of approximating a function:

1. A convergent expansion, or
2. A divergent asymptotic expansion.

A convergent expansion can be easily obtained by integrating term by term the power series representation of the integrand, while a divergent expansion is usually constructed using integration by parts. Depending on the limiting value, one is more favourable than the other.

Recall the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad \mathrm{Re}(z) > 0.$$

One can show using integration by parts that the Gamma function satisfies the functional equation

$$z\Gamma(z) = \Gamma(z+1),$$

which can be used to uniquely extend $\Gamma(z)$ to a meromorphic function on $\mathbb{C}$, with simple poles at the non-positive integers $z = \ldots, -2, -1, 0$. We note that:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt = 2\int_{0^\infty} e^{-u^2}\,du = \sqrt{\pi}.$$

For notational convenience, define the following function:

$$\gamma(z,x) = \int_0^x t^{z-1}e^{-t}\,dt \qquad \text{(Lower incomplete Gamma function)}$$

$$\Gamma(z,x) = \int_x^\infty t^{z-1}e^{-t}\,dt \qquad \text{(Upper incomplete Gamma function)}$$

Lemma 2.1. *For any $n \geqslant 3$,*

$$\int_0^\infty e^{-t^n}\,dt = \Gamma\left(\frac{n+1}{n}\right),$$

where $\Gamma(\cdot)$ is the Gamma function.

Proof. We make a change of variable $u = t^n$, then $du = nt^{n-1}dt = nu^{n-1/n}dt$. The integral becomes:

$$\begin{aligned}\int_0^\infty e^{-t^n}\,dt = \int_0^\infty e^{-u}\left(\frac{du}{nu^{n-1/n}}\right) &= \frac{1}{n}\int_0^\infty u^{-(n-1)/n}e^{-u}\,dt \\ &= \frac{1}{n}\Gamma\left(\frac{1}{n}\right) \\ &= \Gamma\left(\frac{n+1}{n}\right).\end{aligned}$$

■

Example 2.2. Consider approximating the **error function**

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{as } x \longrightarrow \infty.$$

1. **Convergent expansion**
   The Taylor series of the integrand e^{-t^2} around $x = 0$ is
   $$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots,$$
   and it converges everywhere because it has no singularity as a function of complex variable. Integrating the Taylor series term by term, we obtain:
   $$\text{Erf}(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots\right],$$
   and this power series converges everywhere. However, the convergence is slow for large values of x and it doesn't capture the asymptotic behaviour of $\text{Erf}(x)$ as $x \longrightarrow \infty$.

2. **Divergent expansion**
   We first rewrite the error function as follows:
   $$\text{Erf}(x) = \frac{2}{\sqrt{\pi}}\left(\int_0^\infty e^{-t^2}\,dt - \int_x^\infty e^{-t^2}\,dt\right) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt,$$
   Observe that the integrand is almost negligible if $t >> x$, so it contributes most to the integral when t is close to x. Using the identity
   $$e^{-t^2} = -\frac{1}{2t}\frac{d}{dt}\left[e^{-t^2}\right]$$
   and integration by parts, we obtain for every $n \geqslant 0$:
   $$\begin{aligned}G(n) := \int_x^\infty \frac{e^{-t^2}}{t^n}\,dt &= \int_x^\infty \left(-\frac{1}{2t^{n+1}}\right)\left(\frac{d}{dt}[e^{-t^2}]\right)dt\\ &= -\frac{e^{-t^2}}{2t^{n+1}}\bigg|_x^\infty - \int_x^\infty \frac{(n+1)e^{-t^2}}{2t^{n+2}}\,dt\\ &= \frac{e^{-x^2}}{2x^{n+1}} - \left(\frac{n+1}{2}\right)G(n+2)\\ &= \frac{1}{2}\left(\frac{e^{-x^2}}{x^{n+1}} - (n+1)G(n+2)\right)\end{aligned}$$
   Thus,
   $$\begin{aligned}\int_x^\infty e^{-t^2}\,dt = G(0) &= \frac{e^{-x^2}}{2x} - \frac{1}{2}G(2)\\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2(2x^3)} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)G(4)\\ &= \frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2^2x^3} + \frac{(1)(3)e^{-x^2}}{2^3x^5} - \frac{(1)(3)(5)}{2^3}G(6)\end{aligned}$$
   Repeating this procedure using the recurrence relation for $G(n)$, we finally obtain:
   $$\begin{aligned}\text{Erf}(x) &= 1 - \frac{2e^{-x^2}}{\sqrt{\pi}}\left[\frac{1}{2x} - \frac{1}{4x^3} + \frac{1\cdot 3}{8x^5} - \frac{1\cdot 3\cdot 5}{16x^7} + \dots\right]\\ &= 1 - \frac{2e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{N}(-1)^n\frac{(2n-1)!!}{2^{n+1}}\frac{1}{x^{2n+1}} + R_{N+1}(x)\end{aligned}$$

$$= 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \frac{(2n-1)!!}{2^n} \frac{1}{x^{2n+1}} + R_{N+1}(x),$$

where $(2n-1)!! = (2n-1)(2n-3)\dots(3)(1)$ and $R_N(x)$ is the remainder term, having the form

$$R_{N+1}(x) = \frac{(-1)^{N+1}}{\sqrt{\pi}} \frac{(2N+1)!!}{2^{N+1}} G(2(N+1)).$$

Clearly, the series diverges for any large x. For the asymptotic sequence

$$\phi_n(x) = \frac{e^{-x^2}}{x^{2n+1}} \quad \text{as } x \longrightarrow \infty,$$

for a fixed N we have that

$$\begin{aligned}
|R_{N+1}(x)| &= \left| C \int_x^\infty \frac{e^{-t^2}}{t^{2N+2}} \, dt \right| \\
&= \left| C \int_x^\infty \left(-\frac{1}{2t^{2N+3}} \right) \left(\frac{d}{dt} [e^{-t^2}] \right) \, dt \right| \\
&\lesssim C \frac{e^{-x^2}}{x^{2N+3}} \\
&= \frac{C\phi_N(x)}{x^2},
\end{aligned}$$

where C is a constant depending only on N. Hence, we show that

$$\text{Erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{N} (-1)^n \frac{(2n-1)!!}{2^n} \frac{1}{x^{2n+1}} \quad \text{as } x \longrightarrow \infty.$$

To conclude, we present a table with number of terms required for both convergent and divergent expansion to approximate $\text{Erf}(x)$ within 10^{-5}, for different values of x. It clearly suggests that the divergent expansion is a better approximation compare to the convergent expansion.

	Convergent expansion				Divergent expansion	
Range	$x < 1$	$x < 2$	$x < 3$	$x < 5$	$x > 3$	$x > 2.5$
# of terms	8	16	31	75	2	3

Example 2.3. Consider approximating the exponential integral:

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt \quad \text{as } x \longrightarrow \infty.$$

Using integration by parts,

$$\begin{aligned}
E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt &= -\frac{e^{-t}}{t} \bigg|_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2} \, dt = \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2} \, dt \\
&= \frac{e^{-x}}{x} + \frac{e^{-t}}{t^2} \bigg|_x^\infty + 2 \int_x^\infty \frac{e^{-t}}{t^3} \, dt = \frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + 2 \int_x^\infty \frac{e^{-t}}{t^3} \, dt \\
&= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned}$$

$$= e^{-x}\sum_{n=1}^{N}(-1)^{n+1}\frac{(n-1)!}{x^n} + \underbrace{(-1)^N N!\int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt}_{R_N(x)}$$

For the asymptotic sequence

$$\phi_n(x) = \frac{e^{-x}}{x^n}, \quad \text{as } x \longrightarrow \infty,$$

We claim that the first sum is an asymptotic expansion of $E_1(x)$ as $x \longrightarrow \infty$, with respect to the asymptotic sequence $\phi_k(x) = e^{-x}/x^k$. A coarse estimate on the remainder gives:

$$|R_N(x)| \leqslant \frac{N!}{x^{N+1}}\int_x^\infty e^{-t}\,dt = \frac{N!e^{-x}}{x^{N+1}} = \frac{N!}{x}\phi_N(x).$$

Hence,

$$E_1(x) \sim e^{-x}\sum_{k=1}^{N}\frac{(-1)^{k+1}(k-1)!}{k^n} = \sum_{k=1}^{N} a_k\phi_k(x) \quad \text{as } x \longrightarrow \infty.$$

Note that the asymptotic expansion diverges as $N \longrightarrow \infty$ for a fixed x.

We next approximate $E_1(x)$ as $x \longrightarrow 0^+$. Differentiating $E_1(x)$ using Leibniz rule gives:

$$\frac{dE_1}{dx} = -\frac{e^{-x}}{x} = -\frac{1}{x}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right).$$

Integrating term by term, we obtain

$$E_1(x) \sim C - \ln x + x - \frac{x^2}{4} + \dots \quad .$$

To find C, we take the limit as $x \longrightarrow 0^+$:

$$C = \lim_{x\to 0^+}\left[\int_x^\infty \frac{e^{-t}}{t}\,dt + \ln x\right] = -\gamma \approx 0.57772.$$

Note that

$$-\gamma = \Gamma'(1) = \lim_{z\to 0^+}\left[\Gamma z - \frac{1}{z}\right] = -\lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n}\frac{1}{k}\right).$$

Example 2.4. Consider approximating the integral

$$I(x) = \int_0^x t^{-1/2}e^{-t}\,dt = \gamma\left(\frac{1}{2}, x\right) \quad \text{as } x \longrightarrow +\infty.$$

The Taylor series of the integrand $t^{-1/2}e^{-t}$ around $t = 0$ is

$$t^{-1/2}e^{-t^2} = t^{-1/2} - t^{1/2} + \frac{1}{2}t^{3/2} - \frac{1}{6}t^{5/2} + \dots,$$

and it converges for all $t \neq 0$. Integrating this term by term gives:

$$\int_0^x t^{-1/2}e^{-t}\,dt = 2x^{1/2} - \frac{2}{3}x^{3/2} + \frac{1}{5}x^{5/2} - \frac{1}{21}x^{7/2} + \dots,$$

and it doesn't capture the asymptotic behaviour of $\gamma\left(\frac{1}{2}, x\right)$ as $x \longrightarrow \infty$. On the other hand, a direct integration by parts gives:

$$\int_0^x t^{-1/2}e^{-t}\,dt = -t^{-1/2}e^{-t}\Big|_0^x - \frac{1}{2}\int_0^x t^{-3/2}e^{-t}\,dt,$$

which diverges upon evaluating the boundary term at $t = 0$.

To find an expansion that is useful for large x, we rewrite $\gamma\left(\frac{1}{2}, x\right)$ as follows:

$$\begin{aligned}\gamma\left(\frac{1}{2}, x\right) &= \Gamma\left(\frac{1}{2}\right) - \Gamma\left(\frac{1}{2}, x\right)\\ &= \sqrt{\pi} - \int_x^\infty t^{-1/2}e^{-t}\,dt.\end{aligned}$$

Using integration by parts,

$$\begin{aligned}\gamma\left(\frac{1}{2}, x\right) &= -t^{-1/2}e^{-t}\Big|_x^\infty - \frac{1}{2}\int_x^\infty t^{-3/2}e^{-t}\,dt\\ &= x^{-1/2}e^{-x} - \frac{1}{2}\int_x^\infty t^{-3/2}e^{-t}\,dt\\ &= x^{-1/2}e^{-x} + \frac{1}{2}\Big[t^{-3/2}e^{-t}\Big]\Big|_x^\infty + \frac{1\cdot 3}{2^2}\int_x^\infty t^{-5/2}e^{-t}\,dt\\ &= x^{-1/2}e^{-x} - \frac{x^{-3/2}e^{-x}}{2} + \frac{1\cdot 3}{2^2}\int_x^\infty t^{-5/2}e^{-t}\,dt\\ &= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\\ &= \frac{e^{-x}}{\sqrt{x}}\left[1 + \sum_{n=1}^{N}(-1)^n\frac{(2n-1)!!}{(2x)^n}\right] + \underbrace{(-1)^{N+1}\frac{(2N+1)!!}{2^{N+1}}\int_x^\infty t^{-(2N+3)/2}e^{-t}\,dt}_{R_{N+1}(x)},\end{aligned}$$

For the asymptotic sequence

$$\phi_n(x) = \frac{e^{-x}}{x^n x^{1/2}} \quad \text{as } x \longrightarrow \infty,$$

for a fixed N we have that

$$\begin{aligned}|R_{N+1}(x)| = C\int_x^\infty t^{-(2N+3)/2}e^{-t}\,dt &\leqslant \frac{C}{x^{-(2N+3)/2}}\int_x^\infty e^{-t}\,dt\\ &= C\left(\frac{e^{-x}}{x^{(2N+3)/2}}\right)\\ &= C\left(\frac{\phi_N(x)}{x}\right),\end{aligned}$$

where C is a constant depending only on N. Consequently,

$$\Gamma\left(\frac{1}{2}, x\right) \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}}\left[1 + \sum_{n=1}^{N}(-1)^N\frac{(2N-1)!!}{(2x)^N}\right] \quad \text{as } x \longrightarrow \infty.$$

Example 2.5. Consider approximating the integral

$$I(x) = \int_x^\infty e^{-t^4}\,dt \quad \text{as } x \longrightarrow 0^+ \quad \text{and as } x \longrightarrow \infty.$$

For the first limit $x \longrightarrow 0^+$, term-by-term integration of the convergent Taylor series

$$e^{-t^4} = 1 - t^4 + \frac{t^8}{2} - \frac{t^{12}}{3} + \dots,$$

gives a divergent result. In order to make use of this convergent power series, we rewrite $I(x)$ to "remove" the upper bound. From Lemma 2.1, we obtain:

$$\begin{aligned} I(x) = \int_0^\infty e^{-t^4}\,dt - \int_0^x e^{-t^4}\,dt &= \Gamma\left(\frac{5}{4}\right) - \int_0^x e^{-t^4}\,dt \\ &= \Gamma\left(\frac{5}{4}\right) - \int_0^x \left(1 - t^4 + \frac{t^8}{2} - \frac{t^{12}}{3} + \dots\right) \\ &= \Gamma\left(\frac{5}{4}\right) - \left(x - \frac{x^5}{5} + \frac{x^9}{18} - \frac{x^{13}}{36} + \dots\right) \\ &= \Gamma\left(\frac{5}{4}\right) - \sum_{n=0}^{N} (-1)^n \frac{x^{4n+1}}{(4n+1)n!} + R_{N+1}(x), \end{aligned}$$

where $R_{N+1}(x) = o(\phi_N(x))$ as $x \longrightarrow 0^+$ for the asymptotic sequence $\phi_n(x) = x^{4n+1}$. As a result, the leading behavior of $I(x)$ as $x \longrightarrow 0^+$ is

$$\int_x^\infty e^{-t^4}\,dt \sim \Gamma\left(\frac{5}{4}\right) - x \quad \text{as } x \longrightarrow 0^+.$$

Unfortunately, this power series converges slowly for large x. For large values of x, integrating by parts gives:

$$\begin{aligned} \int_x^\infty e^{-t^4}\,dt &= \int_x^\infty \left(-\frac{1}{4t^3}\right)\left(\frac{d}{dt}[e^{-t^4}]\right) dt \\ &= -\frac{e^{-t^4}}{4t^3}\Bigg|_x^\infty - \frac{3}{4}\int_x^\infty \frac{e^{-t^4}}{t^4}\,dt \\ &= \frac{e^{-x^4}}{4x^3} - \frac{3}{4}\int_x^\infty \frac{e^{-t^4}}{t^4}\,dt \end{aligned}$$

One can show that

$$\int_x^\infty e^{-t^4}\,dt \sim \frac{e^{-x^4}}{4x^3} \quad \text{as } x \longrightarrow \infty.$$

Example 2.6. Consider the integral

$$\int_0^\infty e^{-xt^2}\,dt,$$

which has exact value $\frac{1}{2}\sqrt{\frac{\pi}{x}}$. Integrating by parts gives:

$$\int_0^\infty e^{-xt^2}\,dt = \int_0^\infty -\left(\frac{1}{2xt}\right)\left(\frac{d}{dt}[e^{-xt^2}]\right) dt$$

$$= -\frac{e^{-xt^2}}{2xt}\bigg|_0^\infty - \int_0^\infty \frac{e^{-xt^2}}{2xt^2}\,dt,$$

and the boundary term diverges at $t = 0$. It appears that the problem originates from the limits of integration, in which the parameter x appears there.

3 Laplace's Method

Consider the **Laplace integral** which has the form

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt. \tag{3.1}$$

where we assume that f, ϕ are real functions. Note that (3.1) corresponds to the Laplace transform if $\phi(t) = -t$. To investigate the asymptotic behaviour of $I(x)$ as $x \longrightarrow \infty$, we try integrating by parts:

$$\begin{aligned} I(x) &= \int_a^b \left(\frac{f(t)}{x\phi'(t)}\right)\left(\frac{d}{dt}[e^{x\phi(t)}]\right) \\ &= \frac{f(t)e^{x\phi(t)}}{x\phi'(t)}\bigg|_a^b - \underbrace{\int_a^b \frac{e^{x\phi(t)}}{x}\frac{d}{dt}\left(\frac{f(t)}{\phi(t)}\right)dt}_{R(x)} \end{aligned}$$

If $R(x)$ is asymptotically smaller than the boundary term as $x \longrightarrow \infty$, then we have

$$I(x) \sim \frac{f(t)e^{x\phi(t)}}{x\phi'(t)}\bigg|_a^b \quad \text{as } x \longrightarrow \infty. \tag{3.2}$$

In general, (3.2) is satisfied if $f \in C[a,b], \phi \in C^1[a,b]$ and one of the following three conditions holds:

1. $\boldsymbol{\phi'(t) \neq 0}$ **on** $\boldsymbol{[a,b]}$ **and at least one of** $\boldsymbol{f(a), f(b)}$ **are not zero.** These conditions are sufficient to ensure that $R(x)$ exists, and one can show that it becomes negligible compared with the boundary term as $x \longrightarrow \infty$.

2. $\boldsymbol{\phi(t) < \phi(b)}$ **on** $\boldsymbol{[a,b)}$ **and** $\boldsymbol{f(b), \phi'(b) \neq 0}$**.** In this case, $R(x)$ fails to exist but these conditions are strong enough to ensure that

$$I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)} \quad \text{as } x \longrightarrow \infty.$$

3. $\boldsymbol{\phi(t) < \phi(a)}$ **on** $\boldsymbol{[a,b]}$ **and** $\boldsymbol{f(a), \phi'(a) \neq 0}$**.** In this case, $R(x)$ fails to exist but these conditions are strong enough to ensure that

$$I(x) \sim -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)} \quad \text{as } x \longrightarrow \infty.$$

Remark 3.1. A different definition for asymptotic expansion is used in Orszag's book. We say that $f(z) \sim g(z)$ $z \longrightarrow z_0$ if

$$\lim_{z\to z_0} \frac{f(z)-g(z)}{g(z)} = 0.$$

This is equivalent to saying

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = 1 \iff \lim_{z\to z_0} \frac{g(z)}{f(z)} = 1 \iff g(z) \sim f(z) \quad \text{as } z \to z_0.$$

If we write $f(z)$ as

$$f(z) = g(z) + [f(z) - g(z)],$$

$f(z) \sim g(z)$ as $z \longrightarrow z_0$ means that the difference $f(z) - g(z)$ is small compared with $g(z)$ as $z \longrightarrow z_0$. The difference $f(z) - g(z)$ is said to be **subdominant** as compared with $f(z)$ or $g(z)$ which are **dominant**. [Refer to Stokes phenomenon and Example 11, page 116]

Remark 3.2. Assume $t, x \in \mathbb{R}$ in (3.1). If $f(t) = f_{\text{Re}}(t) + i f_{\text{Im}}(t)$, then we look at the real and imaginary parts of the integral separately. If $\phi(t) = \phi_{\text{Re}}(t) + i\phi_{\text{Im}}(t)$, then we rewrite the Laplace integral as:

$$I(x) = \int_a^b \underbrace{f(t)e^{ix\phi_{\text{Im}}(t)}}_{g(x,t)} e^{x\phi_{\text{Re}}(t)}\, dt,$$

where $g(x,t)\colon \mathbb{R}^2 \longrightarrow \mathbb{C}$.

Laplace's method is a general technique for obtaining the asymptotic behaviour as $x \longrightarrow +\infty$ of integrals in which the parameter x appears in an exponential. It is based on the following important observation: if the real continuous function $\phi(t)$ attains its maximum at $c \in [a, b]$ and $f(c) \neq 0$, then only the immediate neighbourhood of $t = c$ that contributes to the full asymptotic expansion of $I(x)$ for large x. That is, we may approximate the integral $I(x)$ by $I(x; \varepsilon)$, where

$$I(x;\varepsilon) = \begin{cases} \displaystyle\int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}\, dt & \text{if } a < c < b, \\ \displaystyle\int_{a}^{a+\varepsilon} f(t)e^{x\phi(t)}\, dt & \text{if } c = a, \\ \displaystyle\int_{b-\varepsilon}^{b} f(t)e^{x\phi(t)}\, dt & \text{if } c = b. \end{cases}$$

Example 3.3. We use Laplace's method to investigate the asymptotic behaviour of

$$I(x) = \int_0^{10} \frac{e^{-xt}}{1+t}\, dt \quad \text{as } x \longrightarrow \infty.$$

Since $\phi(t) = -t$ has a maximum at $t = 0$ over the interval $[0, 10]$, we may replace $I(x)$ by

$$I(x;\varepsilon) = \int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\, dt.$$

Next, we choose $\varepsilon > 0$ sufficiently small such that $(1+t)^{-1} \sim 1$, the first term in its Taylor series about $t = 0$. We then have:

$$I(x) \sim I(x;\varepsilon) \sim \int_0^{\varepsilon} e^{-xt}\, dt = \frac{e^{-xt}}{-x}\bigg|_0^{\varepsilon} = \frac{1 - e^{-\varepsilon x}}{x} \quad \text{as } x \longrightarrow \infty.$$

Since $e^{-\varepsilon x} \ll 1$ as $x \longrightarrow \infty$ for any $\varepsilon > 0$, the leading behaviour of $I(x)$ as $x \longrightarrow \infty$ is

$$I(x) \sim \frac{1}{x} \quad \text{as } x \longrightarrow \infty.$$

Laplace's method also gives the full asymptotic expansion of $I(x)$. For sufficiently small $\varepsilon > 0$, writing $(1+t)^{-1}$ as a geometric series

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n, \quad |t| < 1,$$

we obtain:

$$\begin{aligned} I(x;\varepsilon) = \int_0^{\varepsilon} \frac{e^{-xt}}{1+t}\,dt &= \sum_{n=0}^{\infty} (-1)^n \int_0^{\varepsilon} t^n e^{-xt}\,dt \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\int_0^{\infty} t^n e^{-xt}\,dt - \int_{\varepsilon}^{\infty} t^n e^{-xt}\,dt \right) \end{aligned}$$

The first integral is just the Gamma function:

$$\int_0^{\infty} t^n e^{-xt}\,dt = \frac{1}{x^{n+1}} \int_0^{\infty} s^n e^{-s}\,ds = \frac{\Gamma(n+1)}{x^{n+1}} = \frac{n!}{x^{n+1}},$$

where we make a change of variable $s = xt$. The second integral is subdominant compared with the first integral as $x \longrightarrow \infty$:

$$\int_{\varepsilon}^{\infty} t^n e^{-xt}\,dt = \frac{t^n e^{-xt}}{x}\bigg|_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{n t^{n+1} e^{-xt}}{x}\,dt \sim \frac{\varepsilon^n e^{-\varepsilon x}}{x} \quad \text{as } x \longrightarrow \infty,$$

and this is exponentially smaller than the first integral as $x \longrightarrow \infty$. Hence, the full asymptotic expansion of $I(x)$ as $x \longrightarrow \infty$ is

$$I(x) \sim \sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^{n+1}} \quad \text{as } x \longrightarrow \infty.$$

Recall that the Laplace transform of a given function $f(t)$ is

$$\mathcal{L}(f)(x) = \int_0^{\infty} f(t) e^{-xt}\,dt,$$

and one tries to understand the asymptotic behaviour of $\mathcal{L}(f)(x)$ for large x. Observe that for large x, e^{-xt} is sufficiently small except near $t = 0$, *i.e.* for sufficiently nice functions $f(t)$, the main contribution to $\mathcal{L}(f)(x)$ occurs near $t = 0$. This suggests that we could determine the asymptotic behaviour of $\mathcal{L}(f)(x)$ by approximate $f(t)$ with finitely many terms of its Taylor series around $t = 0$. The rigorous statement of this intuition is called **Watson's lemma**, which is a powerful tool that allows one to explicitly write down an asymptotic expansion of Laplace integrals with only the knowledge of the local behaviour of the integrand.

Theorem 3.4 (Watson's Lemma (Real version))**.** *Consider integrals of the form*

$$I(x) = \int_0^{b} f(t) e^{-xt}\,dt, \quad b > 0,\ b \neq \infty. \tag{3.3}$$

Assume that $f(t) \in C[0,b]$ *and* $f(t)$ *has the asymptotic expansion*

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n} \quad \text{as } t \longrightarrow 0^+, \tag{3.4}$$

with $\alpha > -1$ *and* $\beta > 0$ *so that the integral converges at* $t = 0$*. Then*

$$I(x) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \longrightarrow \infty. \tag{3.5}$$

Proof. We first split $I(x)$ as follows:

$$I(x) = \underbrace{\int_0^{\varepsilon} f(t)e^{-xt}\,dt}_{I(x;\varepsilon)} + \int_{\varepsilon}^{b} f(t)e^{-xt}\,dt,$$

for some $\varepsilon > 0$. The second integral introduces exponentially small errors for any $\varepsilon > 0$:

$$\left|\int_{\varepsilon}^{b} f(t)e^{-xt}\,dt\right| \leqslant \|f\|_{\infty} \int_{\varepsilon}^{b} e^{-xt}\,dt = \|f\|_{\infty}\left(\frac{e^{-\varepsilon x} - e^{-bx}}{x}\right),$$

which converges to 0 exponentially for $x \longrightarrow \infty$. Next, we substitute the asymptotic expansion (3.4) into $I(x;\varepsilon)$ to obtain:

$$I(x;\varepsilon) = \sum_{n=0}^{N}\int_0^{\varepsilon} a_n t^{\alpha+\beta n}e^{-xt}\,dt + \left(I(x;\varepsilon) - \sum_{n=0}^{N}\int_0^{\varepsilon} a_n t^{\alpha+\beta n}e^{-xt}\,dt\right).$$

In particular, we can choose ε sufficiently small such that

$$\left|f(t) - t^{\alpha}\sum_{n=0}^{N} a_n t^{\beta n}\right| \leqslant K t^{\alpha} t^{\beta(N+1)} \quad \text{for every } t \in [0,\varepsilon],$$

for some constant $K > 0$. Thus,

$$\begin{aligned}
\left|I(x;\varepsilon) - \sum_{n=0}^{N}\int_0^{\varepsilon} a_n t^{\alpha+\beta n}e^{-xt}\,dt\right| &\leqslant \int_0^{\varepsilon}\left|f(t) - t^{\alpha}\sum_{n=0}^{N} a_n t^{\beta n}\right| e^{-xt}\,dt \\
&\leqslant K\int_0^{\varepsilon} t^{\alpha+\beta(N+1)}e^{-xt}\,dt \\
&\leqslant K\int_0^{\infty} t^{\alpha+\beta(N+1)}e^{-xt}\,dt \\
&= \frac{K}{x^{\alpha+\beta(N+1)+1}}\int_0^{\infty} s^{\alpha+\beta(N+1)}e^{-s}\,ds \\
&= K\left(\frac{\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta(N+1)+1}}\right) \\
&= \mathcal{O}\left(\frac{1}{x^{\alpha+\beta(N+1)+1}}\right) \quad \text{as } x \longrightarrow \infty.
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_0^{\varepsilon} a_n t^{\alpha+\beta n}e^{-xt}\,dt &= \int_0^{\infty} a_n t^{\alpha+\beta n}e^{-xt}\,dt - \int_{\varepsilon}^{\infty} a_n t^{\alpha+\beta n}e^{-xt}\,dt \\
&= \frac{a_n}{x^{\alpha+\beta n+1}}\int_0^{\infty} s^{\alpha+\beta n}e^{-s}\,ds + \underbrace{\mathcal{O}(e^{-\varepsilon x})}_{\text{as } x\longrightarrow\infty} \qquad \Big[\text{Let } s = xt.\Big] \\
&= \frac{a_n\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} + \underbrace{\mathcal{O}(e^{-\varepsilon x})}_{\text{as } x\longrightarrow\infty}
\end{aligned}$$

Combining all the estimates leads to

$$I(x) - \sum_{n=0}^{N} a_n\frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \sim \mathcal{O}\left(\frac{1}{x^{\alpha+\beta(N+1)+1}}\right) = o\left(\frac{1}{x^{\alpha+\beta(N+1)}}\right) \quad \text{as } x \longrightarrow \infty.$$

The desired result follows since N was arbitrary in the asymptotic representation.

■

Remark 3.5. In the case where $b = \infty$, a far field decay condition is needed to ensure that the integral (3.3) converges, *i.e.* there exists a constant $C > 0$ such that $f(t) \ll e^{Ct}$ as $t \longrightarrow \infty$.

Theorem 3.6 (Watson's Lemma (Complex version))**.** *Suppose* $f(t)$ *is analytic in the sector in the complex plane* $0 < |t| < R, |arg(t)| < \delta < \pi$ *(with a possible branch point at the origin) and suppose*

$$f(t) = \sum_{n=1}^{\infty} a_n t^{\frac{n}{N}-1} \quad \textit{for } |t| < R, \tag{3.6}$$

and

$$|f(t)| \leqslant K e^{bt} \quad \textit{for } R \leqslant t \leqslant T. \tag{3.7}$$

for some K, b *independent of* t*. Then in the sector* $|arg(z)| \leqslant \delta < \frac{\pi}{2}$ *we have*

$$I(z) = \int_0^T f(t)e^{-zt}\,dt \sim \sum_{n=1}^{\infty} a_n \Gamma\left(\frac{n}{N}\right) z^{-n/N} \quad \textit{as } |z| \longrightarrow \infty. \tag{3.8}$$

Proof. Let $z = x + iy$. Similar to the real version, we split $I(z)$ into two parts:

$$I(z) = \int_0^R f(t)e^{-zt}\,dt + \int_R^T f(t)e^{-zt}\,dt = I_1(z) + I_2(z),$$

and estimate each term. We first estimate $I_2(z)$ using (3.7):

$$\begin{aligned} |I_2(z)| \leqslant \int_R^T |f(t)|e^{-xt}\,dt &\leqslant K\int_R^T e^{(b-x)t}\,dt \\ &= K\frac{e^{(b-x)T} - e^{(b-x)R}}{b-x} \\ &= \mathcal{O}(e^{-xR}) \quad \text{as } x \longrightarrow \infty. \end{aligned}$$

Since the power series of $f(t)$ is convergent on the real axis,

$$f(t) = \sum_{n=1}^{M-1} a_n t^{\frac{n}{N}-1} + R_M(t),$$

where the remainder term satisfies

$$R_M(t) \leqslant C t^{\frac{M}{N}-1} \quad \text{for all } 0 < |t| \leqslant R.$$

Thus,

$$\begin{aligned} I_1(z) &= \int_0^R \left(\sum_{n=1}^{M-1} a_n t^{\frac{n}{N}-1}\right) e^{-zt}\,dt + \int_0^R R_M(t)e^{-zt}\,dt \\ &= \underbrace{\sum_{n=1}^{M-1} a_n \left(\int_0^\infty t^{\frac{n}{N}-1}e^{-zt}\,dt\right)}_{J_1(z)} - \underbrace{\sum_{n=1}^{M-1} a_n \left(\int_R^\infty t^{\frac{n}{N}-1}e^{-zt}\,dt\right)}_{J_2(z)} + \int_0^R R_M(t)e^{-zt}\,dt \end{aligned}$$

We can compute $J_1(z)$ and $J_2(z)$ by making a change of variable $s = zt$:

$$J_1(z) = \sum_{n=1}^{M-1} \frac{a_n}{z^{n/N}} \int_0^\infty s^{\frac{n}{N}-1}e^{-s}\,dt = \sum_{n=1}^{M-1} \frac{a_n}{z^{n/N}} \Gamma\left(\frac{n}{N}\right)$$

$$J_2(z) = \sum_{n=1}^{M-1} \frac{a_n}{z^{n/N}} \int_{zR}^{\infty} s^{\frac{n}{N}-1} e^{-s}\, dt = \sum_{n=1}^{M-1} \frac{a_n}{z^{n/N}} \Gamma\left(\frac{n}{N}, zR\right),$$

and the incomplete Gamma function satisfies $\Gamma\left(\frac{n}{N}, zR\right) = \mathcal{O}(e^{-xR})$ as $x \longrightarrow \infty$. Finally,

$$\begin{aligned}
\left|\int_0^R R_M(t)e^{-zt}\, dt\right| &\leqslant C \int_0^R t^{\frac{M}{N}-1} e^{-xt}\, dt \\
&\leqslant C \int_0^\infty t^{\frac{M}{N}-1} e^{-xt}\, dt \\
&= \frac{C}{x^{M/N}} \int_0^\infty s^{\frac{M}{N}-1} e^{-s}\, ds \\
&= \frac{C\Gamma((\frac{M}{N})}{x^{M/N}} \\
&= \mathcal{O}(x^{-M/N}) \quad \text{as } x \longrightarrow \infty.
\end{aligned}$$

Combining all these estimates yields:

$$\begin{aligned}
I(z) &= \sum_{n=1}^{M-1} \frac{a_n}{z^{n/N}} \Gamma\left(\frac{n}{N}\right) + \mathcal{O}(e^{-xR}) + \mathcal{O}(x^{-M/N}) + \mathcal{O}(e^{-xR}) \\
&= \sum_{n-1}^{M-1} \frac{a_n}{z^{n/N}} \Gamma\left(\frac{n}{N}\right) + \mathcal{O}(x^{-M/N}) \quad \text{as } x \longrightarrow \infty.
\end{aligned}$$

The desired result follows since M was arbitrary in the asymptotic representation.

■

Observe that Watson's lemma only applies to Laplace integrals (3.1) with $\phi(t) = -t$. For sufficiently simple $\phi(t)$, we may try a change of variable of the form $s = -\phi(t)$ and obtain:

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\, dt = \int_{-\phi(a)}^{-\phi(b)} -\frac{f(t)}{\phi'(t)} e^{-xs}\, ds = \int_{-\phi(a)}^{-\phi(b)} F(s)e^{-xs}\, ds,$$

where

$$F(s) = -\frac{f(t)}{\phi'(t)} = -\frac{f(\phi^{-1}(-s))}{\phi'(\phi^{-1}(-s))}.$$

Example 3.7. Consider approximating the integral

$$I(x) = \int_0^{\pi/2} e^{-x\sin^2(t)}\, dt \quad \text{as } x \longrightarrow \infty.$$

with $\phi(t) = -\sin^2(t)$. Let $s = -\phi(t) = \sin^2(t)$, then

$$ds = 2\sin(t)\cos(t)dt = 2\sin(t)\sqrt{1-\sin^2(t)}dt = 2\sqrt{s(1-s)}dt.$$

Thus, $I(x)$ transforms into:

$$I(x) = \int_0^1 \frac{e^{-xs}}{2\sqrt{s(1-s)}}\, ds = \frac{1}{2}\int_0^1 F(s)e^{-xs}\, ds,$$

where $F(s) = 1/\sqrt{s(1-s)}$. From the **generalised binomial theorem**,

$$\frac{1}{\sqrt{s(1-s)}} = \frac{1}{\sqrt{s}}\left(\frac{1}{\sqrt{1-s}}\right) = \frac{1}{\sqrt{s}}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}s^n, \quad \text{for } |s|<1.$$

It follows from Watson's lemma (with $\alpha = -1/2, \beta = 1$) that:

$$I(x) \sim \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!\Gamma\left(\frac{1}{2}\right)}\right)\left(\frac{\Gamma\left(-\frac{1}{2}+n+1\right)}{x^{n+1/2}}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2}{n!\Gamma\left(\frac{1}{2}\right)x^{n+1/2}} \quad \text{as } x \longrightarrow \infty.$$

Laplace's Method for Integrals with Movable Maxima

Example 3.8. Consider the Laplace transform of $f(t) = e^{-1/t}$:

$$\mathcal{L}(e^{-1/t})(x) = \int_0^\infty e^{-1/t}e^{-xt}\,dt = \int_0^\infty e^{-\frac{1}{t}-xt}\,dt.$$

Observe that $e^{-1/t}$ vanishes exponentially fast at $t = 0$, the maximum of $\phi(t) = -t$. If we apply Watson's lemma, we obtain the asymptotic series expansion

$$\mathcal{L}(e^{-1/t})(x) \sim 0 \quad \text{as } x \longrightarrow \infty,$$

since the asymptotic power series of $e^{-1/t}$ is 0 as $t \longrightarrow 0^+$. In this case, Watson's lemma does not determine the behaviour of $\mathcal{L}(e^{-1/t})(x)$ since $\mathcal{L}(e^{-1/t})(x)$ is smaller than any power of x as $x \longrightarrow \infty$.

To find the correct behaviour of $\mathcal{L}(e^{-1/t})(x)$, we first determine the location of the maximum of the integrand $e^{-1/t-xt}$, which occurs when:

$$0 = \frac{d}{dt}\left(-\frac{1}{t} - xt\right) = \frac{1}{t^2} - x \implies t = \frac{1}{\sqrt{x}}.$$

Such a maximum is called a movable maximum because its location depends on the parameter x. To deal with this, we make a change of variable to transform the movable maximum to a fixed maximum. Let $t = \dfrac{s}{\sqrt{x}}$, then

$$\mathcal{L}(e^{-1/t})(x) = \int_0^\infty e^{-\frac{1}{t}-xt}\,dt = \frac{1}{\sqrt{x}}\int_0^\infty e^{-\sqrt{x}\left(s+\frac{1}{s}\right)}\,ds$$

Example 3.9. Consider approximating the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \quad \text{as } x \longrightarrow \infty.$$

This is a Laplace integral, with $f(t) = e^{-t}/t$ and $\phi(t) = \ln t$. Laplace's method is not immediately applicable here since

$$\max_{t\in[0,\infty)} \phi(t) = \infty,$$

and the maximum occurs as $t \longrightarrow \infty$ where $f(t)$ is exponentially small. Neglecting the term $1/t$ which vanishes algebraically as $t \longrightarrow \infty$, we determine the location of the maximum of $t^x e^{-t}$, which occurs when:

$$0 = \frac{d}{dt}(t^x e^{-t}) = t^x(-e^{-t}) + (xt^{x-1})e^{-t} \implies t^x = xt^{x-1} \implies t = x.$$

Since this is a movable maximum, we make a change of variable $t = sx$ and obtain:

$$\begin{aligned}
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt &= x^x \int_0^\infty s^{x-1} e^{-sx}\, ds \\
&= x^x \int_0^\infty \frac{e^{x \ln s} e^{-xs}}{s}\, ds \\
&= x^x \int_0^\infty \frac{e^{x(\ln s - s)}}{s}\, ds.
\end{aligned}$$

4 Problems

1. Show that if

$$f(x) \sim a(x - x_0)^{-b} \quad \text{as } x \longrightarrow x_0^+,$$

then

$$\int_{x_0}^{x} f(x)\, dx \sim \frac{a}{1-b}(x - x_0)^{1-b} \quad \text{as } x \longrightarrow x_0^+ \quad \text{if } b < 1.$$

Solution:

2. (a) Give an example of an asymptotic relation $f \sim g$ as $x \longrightarrow \infty$ that cannot be exponentiated, *i.e.* $e^{f(x)} \sim e^{g(x)}$ as $x \longrightarrow \infty$ is false.

Solution:

(b) Show that if $f(x) - g(x) \ll 1$ as $x \longrightarrow \infty$, then $e^{f(x)} \sim e^{g(x)}$ as $x \longrightarrow \infty$.

Solution:

3. Find the leading behaviour as $x \longrightarrow 0^+$ of the following integrals.

(a) $\displaystyle\int_0^1 e^{-x/t}\, dt;$

Solution:

(b) $\displaystyle\int_x^1 \cos(xt)\, dt;$

Solution:

(c) $\displaystyle\int_0^{1/x} e^{-t^2}\, dt;$

Solution:

(d) $\displaystyle\int_1^{\infty} \frac{\cos(xt)}{t}\, dt.$

Solution:

4. Consider

$$I(x) = \int_0^{\infty} \frac{e^{-t}}{1 + xe^{t^2}}\, dt.$$

(a) Show that $I(x) - 1 \sim -\exp\left(\sqrt{-\ln x}\right)$ as $x \longrightarrow 0^+$.

Solution:

(b) Find the full asymptotic expansion of $I(x)$ as $x \longrightarrow 0^+$.

Solution:

5. Use Laplace's method to determine the leading behaviour of the following integrals.

   (a) $\int_0^{\pi/2} \sqrt{t} e^{-x \sin^4 t}\, dt$ as $x \longrightarrow \infty$;

   Solution:

   (b) $\int_0^1 \sqrt{\tan t} e^{-xt^2}\, dt$ as $x \longrightarrow \infty$.

   Solution:

6. Use Watson's lemma to obtain an asymptotic expansion of the exponential integral

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

Hint: Show that

$$E_1(x) = e^{-x} \int_0^\infty \frac{e^{-xt}}{1+t}\, dt.$$

Solution:

7. The modified Bessel function $I_n(x)$ has the integral representation

$$I_n(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos\theta} \cos(n\theta)\, d\theta.$$

Show that $I_n(x) \sim \dfrac{e^x}{\sqrt{2\pi x}}$.

Solution: